Gevrey Classes

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0.1 Introduction example-Euler-Ode

Consider

$$x^2 f'(x) + f(x) = x$$

Trying to solve for $\widetilde{f}(x) = \sum f_n x^n$, we find

$$(n-1)\, f_{n-1} + f_n = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

So

$$f_n = (-1)^n (n-1)! \quad \Rightarrow \quad f \sim \sum (-1)^n \, n! x^{n+1}$$

Consider

$$F(z) = \sum (-1)^n z^n = \frac{1}{1+z}$$

then formally

$$\int_0^\infty F(xt) e^{-t} dt = \sum (-1)^n x^n \int_0^\infty t^n e^{-t} dt \sim \sum (-1)^n n! x^n$$

so

$$f(x) = x \int_0^\infty F(xt) e^{-t} dt = \int_0^\infty \frac{xe^{-t}}{1+xt} dt = \int_0^\infty \frac{1}{1+t} e^{-t/x} dt$$

is a good candidate for solution, indeed

$$x^2 f^{'}(x) + f(x) = \int_0^\infty \frac{1}{1+t} \left[\frac{x^2 t}{x^2} + 1 \right] e^{-t/x} dt = \int_0^\infty e^{-t/x} = x.$$

we can analytically continued this f in $|\arg(z)| < \pi$, and by passing through the negative ray, we get a new solution

$$f(x) + Ce^{\frac{1}{x}}$$

When does it works? why it works? etc...

0.2 Borel-Laplace summation

Consider a formal power series

$$\widetilde{a}(x) = \sum_{n\geq 0}^{\infty} a_n x^n$$

such that

$$|a_n| \leq C^{n+1} n!$$

We define its Borel transform transform

$$A(z) = (\mathcal{B}a)(z) = \sum_{n\geq 0}^{\infty} \frac{a_n}{n!} x^n$$

It has positive radius of convergence. If A has an analytic continuation to a NH of the positive ray, we defined the Laplace transform as

$$(\mathcal{L}A)(x) = \int_0^\infty A(xt)e^{-t}dt = \frac{1}{x}\int_0^\infty A(u)e^{-\frac{u}{x}}du \sim \widetilde{a}(x)$$

If a is such that $a = \mathcal{L}\mathcal{B}\widetilde{a}$,we say that $\widetilde{a}$ is Gevrey summable to a.

Properties of Gevrey summation:

1. Linear
2. Commute with $x \mapsto cx$ and therefore with its infinitesimal generator $E = x\frac{d}{dx}$ ($Ex^n = nx^n$diagonal)
3. Agree with regular sums.
4. at infinity changing $a(x) = \frac{1}{x}a(\frac{1}{x})$,or considering $f(x) = \int_0^\infty A(u)e^{-ux}du = \mathcal{L}_\infty A$ it diagonalize shifts

   $$f(x+t) = \mathcal{L}_\infty \left\{e^{-tu}A(u)\right\}(x)$$

   and therefore also derivatives.

we are after a class of function such that the Borel Laplace Scheme works, and uniquely reminder the function

0.3 Gevrey functions in a sector

Consider the sector

$$S(\theta, r) = \{z : |\arg z < \theta|, |z| < r\}, \quad \theta > 0$$

We define the class $\mathcal{G}_\theta$ (at the origin) as the class of all functions, f, analytic in $S(\theta, r)$ for some $r > 0$, and in every closed sub sector it is C^∞ including the origin and

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} + R_N(z)$$

where $|R_N(z)| \leq C_{\theta'}^{n+1} n! |z|^n$ for any $|\theta'| < \theta$.

Some properties

1. it follows that any every function $\left|\frac{f^{(n)}(z)}{n!}\right| \leq C_{\theta'}^{n+1} n!$
2. vector space
3. definable algebra
4. closed under composition
5. division by coordinate if $f \in \mathcal{G}_\theta$ and f has a zero of order n at the origin.
6. quasianalyticty if $\theta > \frac{\pi}{2}$ (i.e. if $f \in \mathcal{G}_\theta$. and $f \sim_{\mathcal{G}} 0$ then $f \equiv 0$)
7. Interpolation if $\theta < \frac{\pi}{2}$, if $|a_n| \leq C^{n+1} n!$,there is $f \in \mathcal{G}_\theta$ such that $f \sim_{\mathcal{G}} \sum a_n z^n$ (Borel-Ritt Lemma)

Difference then analytic category:

1. Formal solution does not grantee a solution. If $f(x^k) = g(x)$, with $g \in \mathcal{G}_\theta$ and $f \in C^\infty$, then not necessarily $f \in \mathcal{G}_\theta$. Indeed say $k = 2$ and

$$g(x) \sim \sum a_{2n} x^{2n}$$

.with $a_n \asymp n!$, then

$$f(x) \sim \sum a_{2n} x^n$$

with

$$a_{2n} \asymp (2n)! \asymp n!^2$$

In dim=2 even worst when we consider blowups

2. Not known if Noetherian ring, Not known if divisible property holds, (say we know we can decide in C^∞ the polynomial $x^3 + y^2$ is it true in $\mathcal{G}_\theta$

0.4 Summation and Structure of Gevrey classes.

Consider the class of function A_θ consists of analytic functions, F, in at zero $\mathbb{R}_+ \cup \{z : |\arg z| < \theta, |z| > R\}$ and satisfies in any closed sub-sector

$$|F(z)| \leq C e^{|z|B}, \quad |\arg z| \leq \theta' < \theta$$

We have the following characterization theorem

Theorem (Nevanlina-Sokal): $\mathcal{L} : A_\theta \to G_{\theta+\pi/2}$ is a bijection with inverse $\mathcal{B}$, moreover in $G_{\theta+\pi/2}$, $\mathcal{B}$ can be realized as the integral transform

$$\mathcal{B}F(z) = \frac{1}{2\pi i} \int_{\partial S(\theta', r)} f(t) e^{\frac{t}{z}} \frac{dt}{t}$$

and in $G_{\theta+\pi/2}$, any function is Gervey summable

Theorem (extended Watson Lemma): If $\lambda > -1$, then $\mathcal{L} : z^\lambda A_\theta(z) \to t^\lambda G_{\theta+\pi/2}(t)$ is a bijection

0.5 Saddle point example- the gamma function

Let consider another example that raises a gervey function
$\Gamma(z+1) = \int_0^\infty t^z \exp(-t)dt$

Claim: $\Gamma(z+1) = z^{z+1}e^{-z}\frac{1}{\sqrt{z}}f(z)$,where $f \in \mathcal{G}_\pi^\infty$

Proof: change of variable $t = uz$, we get

$$\Gamma(z+1) = \int_0^\infty (uz)^z \exp(-uz)zdu = z^{z+1}\int_0^\infty e^{z(\log u - u)}du$$
$$= z^{z+1}e^{-z}\int_0^\infty e^{z(\log u - u + 1)}du = z^{z+1}e^{-z}\int_0^\infty e^{zg(u)}du$$

we first do the change of variable $g(u) = -t^2$,

$$\frac{1}{u} - 1du = -2t^2dt$$

$$\int_0^\infty e^{zg(u)}du = \int_{-\infty}^\infty e^{-zt^2}\frac{2u(t)tdt}{u(t)-1}$$

then we map $t^2 = w$ and get

$$\int_0^\infty e^{zg(u)}du = \int_0^\infty \frac{1}{\sqrt{w}}e^{-zw}\left[\frac{u(\sqrt{w})\sqrt{w}dt}{u(\sqrt{w})-1} - \frac{u(\sqrt{-w})\sqrt{-w}dt}{u(\sqrt{-w})-1}\right]$$

It can be seen that

$$\frac{u(\sqrt{w})\sqrt{w}dt}{u(\sqrt{w})-1} - \frac{u(\sqrt{-w})\sqrt{-w}dt}{u(\sqrt{-w})-1} \in A_{\pi/2}$$

so

$$\int_0^\infty e^{zg(u)}du \in \frac{1}{\sqrt{z}}G_\pi^\infty$$

Remark: We can write it differently via Cauchy formula the change $\int_0^\infty e^{zg(u)}du = \int_0^\infty e^{-zw}q(w)dw$, we will get

$$q(w) = \frac{1}{\sqrt{w}}\frac{1}{2\pi i}\int_\Gamma \frac{\sqrt{g(s)}}{g(s)-w}ds$$

where Γ is a NH of $[0,\infty)$.